Semilinear Wave Equations on Decelerated Expanding FLRW Spacetimes

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Abstract

In this dissertation we aim to present in detail the proof of blow-up for solutions of an initial value problem associated to a family of quasilinear wave equations in $3+1$ dimensions following [FJ85] by Fritz John. Employing techniques used in the proof of this result, we attempt to show blow-up in 3+1 dimensions for solutions to the semilinear wave equation[1] $\Box u = -\left[2p/\left((1-p)(t+1)\right)\right] u_t + (u_t)^2$, where p is a fixed parameter satisfying $p \in (0,1)$. The core argument of our (tentative) proof involves a continuity argument over solutions of the spherical average of this PDE, at $p = 0$, together with integral estimates leading to an ODE type inequality along certain characteristics.

1 Introduction

The initial value problem for nonlinear wave equations has been studied quite extensively, particularly the semilinear and quasilinear cases, see e.g. [Sog95], [Ringström], [Luk] and references therein. The main questions surrounding this topic concern *well-posedness* i.e., local existence, uniqueness and stability under continuous perturbations of the initial data, and those of *global existence* i.e., long time behavior of solutions, whether they are defined for all time, within a certain function space or, on the contrary, if they *blow-up in finite time*, i.e. their norm in that function space goes to infinity.

The general object of study is the following Cauchy problem for semilinear and quasilinear wave equations

$$\begin{cases} \Box u := \partial_t^2 u - \Delta u = \phi(x, t, u, u', u''), \\ u(x, 0) = f(x), \;\; u_t(x, 0) = g(x), \;\; x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is the solution, u' and u'' represent, respectively, the vector of all first order derivatives of u and the vector of all second order derivatives of u, where ϕ is a nonlinearity, and f, g, the initial data satisfying certain assumptions, mainly having compact support and appropriate regularity.

[1] We consider the d'Alembert operator $\Box$ (without a lower metric index) to have the signature of (1.1), following the works by Fritz John.

We further assume that, on linearizing (1.1) in a neighbourhood of the trivial solution $u = 0$, one recovers the classical wave equation, which implies in particular that

$$\phi = \frac{\partial \phi}{\partial u} = 0, \quad \frac{\partial \phi}{\partial u'} = 0, \quad \frac{\partial \phi}{\partial u''} = 0, \text{ at } (x, t, 0, 0, 0). \tag{1.2}$$

Local-well posedness of (1.1) is well established for fairly general conditions on ϕ, f and g, see Theorem 2.1 in the following section, [Sog95]. The long time behavior of solutions to (1.1) however depends significantly on the type of nonlinearity, properties of the initial data and, perhaps more surprisingly, on the spatial dimension n. In particular, for $n \geq 4$, there are certain types of non-nonlinearities ϕ for which it is known that (1.1) admits global in time solutions for sufficiently small initial data, Theorem 2.3 in the following section, [Sog95]. For $n = 3$ however, for a family of nonlinearities, which include some of the type previously considered for $n \geq 4$, Theorem 2.4 shows that blow-up occurs independently of the size of the initial data, provided this initial data is non-trivial, [FJ85]. A detailed description of the proof of this result is a significant part of the thesis, see Theorem 2.4 in the following section for a brief overview and Chapter 4 of thesis for the full proof.

Nonlinear wave equations feature prominently in mathematical relativity, typically arising in the context of the Einstein equations by fixing local (wave) coordinates, where the solution is a metric $\boldsymbol{g}$, see for example Section 4.1 in [Chr20]. In this work we fix a Lorentzian manifold $\mathcal{M}$, endowed with a certain Lorentzian metric $\boldsymbol{g}$, and study the wave equation in this background for a scalar function u defined on this manifold. In particular, the wave operator $\Box_{\boldsymbol{g}}$ (d'Alembertian) has the following form in local coordinates

$$\Box_{\boldsymbol{g}} = \frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\alpha (\boldsymbol{g}^{\alpha\beta} \sqrt{|\boldsymbol{g}|} \partial_\beta), \tag{1.3}$$

for $\alpha, \beta \in \{0, 1, ..., n\}$, see for example [Lee19] on how to obtain this. In this work we simply consider the manifold $\mathbb{R}^3 \times \mathbb{R}_+$, with global coordinates (t, x^1, x^2, x^3), equipped with a metric of the form[2]

$$\boldsymbol{g} := -dt^2 + t^{2p} \sum_{i=1}^{3} (dx^i)^2. \tag{1.4}$$

In this context, we consider an initial value problem associated to the following wave equation

$$\Box_{\boldsymbol{g}} u = -(u_t)^2, \tag{1.5}$$

where $\Box_{\boldsymbol{g}}$ depends on the value of p. We now make the following observations:

- For $p = 0$, corresponding to a flat and static spacetime (Minkowski's Spacetime), equation (1.5) leads to

$$u_{tt} - \Delta u = (u_t)^2. \tag{1.6}$$

  This is a well known example for which the blow-up result in $3 + 1$ dimensions by Fritz John holds, see Example 2.2 in the following section.

- For $p > 1$, corresponding to a universe in accelerated expansion, Theorem 2.1 and Remark 2.2 in [CFO23] show, as a special case, that (1.5) admits global non-trivial solutions for sufficiently small initial data, as a consequence of the accelerated character of the expansion.

[2] Here, we consider a different metric signature compared to (1.1), following [CFO23].

- For $p \in (0,1)$, corresponding to a universe in decelerated expansion, it is unknown (to the author's current knowledge) whether solutions of (1.5) exist globally in time or, on the contrary, if they blow-up in finite time. In this setting, (1.5) leads to

$$\Box u = -\frac{2p}{(1-p)(t+1)} u_t + (u_t)^2, \tag{1.7}$$

  which corresponds to our main object of study in this dissertation in the context of an initial value problem.

Our general goal is to prove blow-up in finite time for solutions of (1.7) and understand its dependence on the parameter p.

2 Nonlinear Wave Equation

2.1 Local existence

The following theorem (Theorems 4.1 and 4.3 in Chapter I of [Sog95]) establishes well-posedness of the IVP (1.1) for small deviations from Minkowski's metric, or, equivalently, small deviations from the standard linear wave operator. Furthermore, it provides a breakdown criteria, i.e. obstructions to global existence in time.

Theorem 2.1. *Consider the following initial value problem*

$$\begin{cases} \sum_{\alpha,\beta=0}^{n} g^{\alpha\beta}(u,u')\partial_\alpha\partial_\beta u = F(u,u'), \\ u(x,0) = f(x), \;\; u_t(x,0) = g(x), \;\; x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

for $0 \le \alpha, \beta \le n$, *where* $g^{\alpha\beta}$ *and* F *are* C^∞ *with all derivatives* $O(1)$ *and* $f, g \in C_0^\infty(\mathbb{R}^n)$. *Assume also that* $F(0,0) = 0$ *and*

$$\sum_{\alpha,\beta} |g^{\alpha\beta} - g_0^{\alpha\beta}| < \frac{1}{2}, \tag{2.2}$$

where $g_0^{\alpha\beta} = diag(1, -1, ..., -1)$. *Then there is a* $T > 0$ *such that (2.1) has a unique solution* $u \in C^\infty([0,T] \times \mathbb{R}^n)$. *If* T_* *denotes the supremum of such times* T, *then either* $T_* = \infty$ *or*

$$\sum_{|k| \le (n+6)/2} \left|\partial^k u(x,t)\right| \notin L^\infty([0,T_*) \times \mathbb{R}^n). \tag{2.3}$$

Remark 2.1. *This result can be generalized for initial conditions in certain Sobolev spaces, in particular for* $(f,g) \in H^{s+1} \times H^s$ *with* $s \ge n+2$, *see Theorem 4.1 in [Sog95].*

2.2 Domain of dependence

The following results (Theorems 4 and 4a of [FJ85]) show that sufficiently regular solutions of (1.1) in $n+1$ dimensions have finite speed of propagation. Furthermore, it defines a domain of dependence similar to the linear case, for sufficiently regular solutions.

Theorem 2.2. *Let $\phi(x,t,u,u',u'')$ be a C^2 function of its arguments satisfying*

$$\phi(x,t,0,0,u'') = 0,\ \forall\, x,t,u''. \tag{2.4}$$

Let u be a C^2 solution of the following equation

$$\Box u = \phi(x,t,u,u',u''), \tag{2.5}$$

in the cone

$$K(x_0,a) = \{(x,t) \,|\, \|x-x_0\| + t \le a;\ t \ge 0\}, \tag{2.6}$$

for some $x_0 \in \mathbb{R}^n$, $a \in \mathbb{R}_+$, with initial conditions

$$u(x,0) = u_t(x,0) = 0,\ \textit{for}\ \|x-x_0\| < a. \tag{2.7}$$

Then $u(x,t) = 0$ in the cone $K(x_0,a)$.

In particular, we have the following corollary to Theorem 2.2

Corollary 2.2.1. *Assume ϕ is in the conditions of Theorem 2.2 and let u be a C^2 solution of*

$$\Box u = \phi(x,t,u,u',u''), \tag{2.8}$$

in the slab

$$x \in \mathbb{R}^n,\ 0 \le t < T. \tag{2.9}$$

Assume further that

$$u(x,0) = u_t(x,0) = 0,\ \textit{for}\ \|x\| \ge R. \tag{2.10}$$

Then

$$u(x,t) = 0,\ \textit{for}\ \ 0 \le t < T,\ \|x\| \ge R+t. \tag{2.11}$$

This is a crucial result used in the blow-up proof for quasilinear wave equations in $3+1$ dimensions.

2.3 Global existence

In regards to global existence, the following theorem states that solutions to certain families of quasilinear wave equations in $n+1$ dimensions can be extended globally, if $n \ge 4$, for sufficiently small initial data (Theorem 2.1 in Chapter II of [Sog95]).

Theorem 2.3. *Let $n \ge 4$ and consider the following initial value problem*

$$\begin{cases} \sum_{\alpha,\beta=0}^{n} g^{\alpha\beta}(u')\partial_\alpha\partial_\beta u = F(u'), \\ u(x,0) = \epsilon f(x),\ \ u_t(x,0) = \epsilon g(x),\ \ x \in \mathbb{R}^n, \end{cases} \tag{2.12}$$

for $0 \le \alpha, \beta \le n$, where $g^{\alpha\beta}$ and F are C^∞ and $f, g \in C_0^\infty(\mathbb{R}^n)$. Assume also that

$$\sum_{\alpha,\beta=0}^{n} g^{\alpha\beta}(0)\partial_\alpha\partial_\beta = \Box,\ \ F(0) = 0,\ \ F'(0) = 0. \tag{2.13}$$

Then, there exists a sufficiently small $\epsilon > 0$ such that (2.12) admits a global C^∞ solution.

As mentioned in the Introduction, Theorem 2.3 does not extend to the $n = 3$ case, at least for some nonlinearities ϕ.

2.4 Blow-up for quasilinear wave equations in 3+1 dimensions

We have the following theorem by Fritz John (Theorem 1 of [FJ85])

Theorem 2.4. *Let $n = 3$ and let $u \in C^2(\mathbb{R}^3 \times \mathbb{R}_+)$ be a solution of*

$$\Box u(x,t) = \phi(x,t,u,u',u''), \tag{2.14}$$

with initial data $u|_{t=0}$, $u_t|_{t=0} \in C_0^2(\mathbb{R}^3)$ and where

$$\phi(x,t,u,u',u'') = A(x,t,u,u',u'') + \frac{\partial}{\partial t}\left(B(x,t,u,u')\right), \tag{2.15}$$

for functions $A \in C^2$, $B \in C^3$, satisfying the following properties

$$A(x,t,u,u',u'') \geq 0, \ \ \forall x,t,u,u',u'', \tag{2.16}$$

$$A(x,t,0,0,u'') = 0, \ \ \forall x,t,u'', \tag{2.17}$$

$$\exists\, (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\} \ s.t. \ B(x,t,u,u') \geq (au + bu_t)^2, \ \ \forall x,t,u,u', \tag{2.18}$$

$$B(x,t,0,0) = 0, \ \ \forall x,t. \tag{2.19}$$

Assuming further that

$$K := \int_{\mathbb{R}^3} \big(u_t(x,0) - B(x,0,u(x,0),u'(x,0))\big)dx \geq 0, \tag{2.20}$$

then $u \equiv 0$.

Under the assumptions, this result shows that a global C^2 solution of (2.15) must be the trivial one. As a consequence, if one sets non-trivial initial data, the solution cannot exist for all time $t \geq 0$, implying blow-up in finite time. The proof of this result relies on Theorem 2.2 and depends significantly on whether $K = 0$ or $K > 0$. Before describing the ideas used in the proof we first provide two examples, Examples 2.1 and 2.2, showing this dichotomy in K.

Example 2.1. *Let $u \in C^2$ be a solution of the following IVP*

$$\begin{cases} \Box u = 2u_t u_{tt}, \\ u(x,0) = f(x), \ \ u_t(x,0) = g(x), \ \ x \in \mathbb{R}^3, \end{cases} \tag{2.21}$$

where $f, g \in C_0^2(\mathbb{R}^3)$, and assuming that

$$K = \int_{\mathbb{R}^3} \left(g(x) - g^2(x)\right) dx > 0. \tag{2.22}$$

This example satisfies the assumptions of Theorem 2.4 and, via its proof, one can find an upper bound for the time of blow-up given by

$$t^* = 2R \exp\left(\frac{64\pi R^3}{3K}\right), \tag{2.23}$$

along the characteristic $(r, r+R)$, with $r > R$, where R is the radius of a ball centered at 0 containing the compact support of the initial data, which we denote by $B(0,R)$. Note that blow-up of u in this example occurs also for $K = 0$, since Theorem 2.4 holds for $K \geq 0$. This example can be seen in more detail in [FJ90].

Remark 2.2. *The upper bound on the time of blow-up (2.23) increases with R. Thus, the smaller the support of the initial data, the sooner we expect blow-up of the solution to happen. Interestingly, there is an almost global existence result in* $3+1$ *dimensions which shows, under certain assumptions, that the time of existence of solutions increases exponentially with an* ϵ *factor in the initial data, as considered in (2.12), see Theorem 2.2 of Chapter II in [Sog95].*

We have the following corollary to Theorem 2.4 (Theorem 2 of [FJ85])

Corollary 2.4.1. *Let* $n = 3$ *and consider* $C(x,t,v,v',v'') \in C^3$ *on all its arguments. Assume further that*

$$\exists\, (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\} \ s.t. \ C(x,t,v,v',v'') \geq (av_t + bv_{tt})^2, \ \ \forall x,t,v,v',v'', \tag{2.24}$$

$$C(x,t,v,v',v'') = 0, \ for \ v_t = v'_t = 0. \tag{2.25}$$

If $v \in C^3(\mathbb{R}^3 \times \mathbb{R}_+)$ *is a solution of*

$$\Box v = C(x,t,v,v',v''), \tag{2.26}$$

with v, v_t *and* v_{tt} *of compact support at* $t = 0$*, then* $v \equiv 0$.

Blow-up for the following example (Example 2.2), corresponding to the initial value problem associated to equation (1.6), can be obtained via the above corollary.

Example 2.2. *Let* $v \in C^3$ *be a solution of the following IVP*

$$\begin{cases} \Box v = (v_t)^2, \\ v(x,0) = f(x), \ \ v_t(x,0) = g(x), \ \ x \in \mathbb{R}^3, \end{cases} \tag{2.27}$$

where $f, g \in C_0^2(\mathbb{R}^3)$*. Choosing* $a = 1$ *and* $b = 0$ *in (2.24) the assumption is satisfied. If* $v_t = 0$ *then* $C = 0$ *and (2.25) is also satisfied. Thus, from Theorem 2.4.1, we conclude that* $v \equiv 0$*. This implies, when considering (2.27) with non-trivial initial data, that* v *must blow-up in finite time. If one lets* $u = v_t$ *then*

$$\Delta v = v_{tt} - C = u_t - C =: u_t - B, \tag{2.28}$$

and we obtain from (2.20)

$$\begin{aligned} K &= \int_{\mathbb{R}^3} \big(u_t(x,0) - B(x,0,v(x,0),u(x,0),v_i(x,0),v_{tt}(x,0),u_i(x,0),v_{ij}(x,0))\big)dx \\ &= \int_{\mathbb{R}^3} \Delta v(x,0)dx = \int_{\mathbb{R}^3} \nabla \cdot \nabla v(x,0)dx. \end{aligned} \tag{2.29}$$

Since v *has compact support at* $t = 0$*, using the divergence theorem and considering a compact region* V *containing the support of* $v|_{t=0}$*, we obtain*

$$K = \int_{\mathbb{R}^3} \nabla \cdot \nabla v(x,0)\, dx = \int_V \nabla \cdot \nabla v(x,0)\, dx = \int_{\partial V} \nabla v(x,0) \cdot \hat{n}\, dS = 0. \tag{2.30}$$

Thus, $K = 0$ *in this particular case.*

Remark 2.3. *Note that, for* $n \geq 4$*, equation (2.27) is in the setting of Theorem 2.3, where we have global existence of the solution provided that the initial data is sufficiently small. Thus, dimension has a significant influence in the long time behavior of solutions to nonlinear wave equations.*

We now briefly enumerate the main ideas behind the proof of Theorem 2.4 using (2.27) as a guiding example. The general statement, Theorem 2.4, follows the same ideas with some extra considerations allowing for the nonlinearity given by (2.15).

1. Let v be a global C^2 solution of (2.27). The initial value problem falls within the setting of the uniqueness result, Corollary 2.2.1. From it, we obtain

$$v(x,t) = 0, \text{ for } \|x\| > t + R, \ \ t \geq 0,$$

where R corresponds to the radius of a ball containing the compact support of the initial data.

2. Let $\overline{v}(r,t)$ denote the spherical average of $v(r,t)$, extended symmetrically in the spatial dimension for all $\mathbb{R}$. Then, using the Euler-Poisson-Darboux equation (see e.g. [Evans10]), the spherical average of $\Box v = v_t^2$ leads to the 1+1-dimensional wave equation $\Box(r\overline{v}) = r\overline{v_t^2}$. By Duhamel's principle (see again [Evans10]), this equation has the following solution

$$\overline{v}(r,t) = \overline{v}^0(r,t) + \int\int_{T_{r,t}} \frac{\rho}{2r}\overline{v_t^2}(\rho,\tau)d\rho d\tau,$$

where $T_{r,t}$ is the characteristic triangle with vertex (r,t).

3. Fixing $(r,t) \in \Sigma$ from now on, where $\Sigma = \{(r,t)\,|\, r + R < t < 2r\}$, by symmetry considerations and the fact that $\overline{v}^0(r,t) = 0$ for $(r,t) \in \Sigma$, we can write the above equation as

$$\overline{v}(r,t) = \int\int_{T^*_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau, \ \ (r,t) \in \Sigma, \tag{2.31}$$

where $T^*_{r,t}$ is the trapezoid with vertices $\{(r,t),(0,t-r),(t-r,0),(t+r,0)\}$.

4. Letting $c := t - r$, which satisfies $R < c < r$, we consider another region $S_{r,t} \subset T^*_{r,t}$ such that, by applying Cauchy-Schwarz's inequality to (2.31), we obtain

$$\overline{v}(r,t) \geq \int\int_{S_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau \geq \int_c^r \int_{\rho-R}^{\rho+c} \frac{\rho}{2r}\overline{v_\tau}^2(\rho,\tau)d\tau d\rho, \ \text{ for } \ R < c < r. \tag{2.32}$$

For a sketch of the regions under consideration see the following figure, Fig. (1).

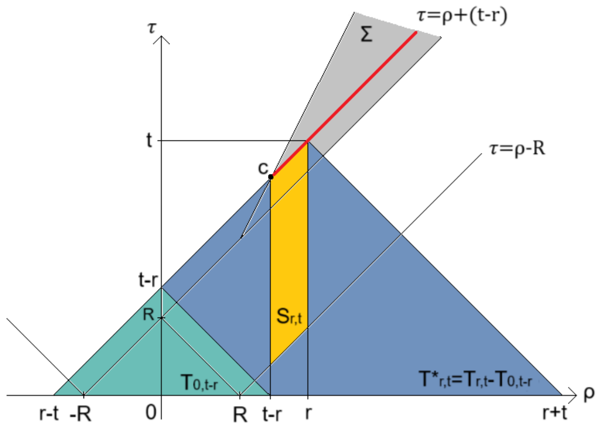

Figure 1: The characteristic c (in red) and the regions $T^*_{r,t} = T_{r,t} \setminus T_{0,t-r}$ (blue), Σ (gray) and $S_{r,t}$ (yellow).

5. In the $S_{r,t}$ region, since $v \in C^2$ and $\overline{v}(\rho,\tau) = 0$ for $\tau \leq \rho - R$, we have $\overline{v}(\rho,\rho-R) = \overline{v}_\tau(\rho,\rho-R) = 0$ and thus
$$\overline{v}(\rho,\rho+c) = \int_{\rho-R}^{\rho+c} \partial_\tau \overline{v}(\rho,\tau)d\tau = \int_{\rho-R}^{\rho+c} \overline{v_\tau}(\rho,\tau)d\tau.$$
Using again Cauchy-Schwarz's inequality in the above equation we obtain the following relation
$$\frac{1}{\Gamma(c)}\overline{v}^2(\rho,\rho+c) \leq \int_{\rho-R}^{\rho+c} \overline{v_\tau}^2(\rho,\tau)d\tau, \tag{2.33}$$
where $\Gamma(c) = c + R > 0$.

6. Using (2.32) and (2.33), along the characteristic $(r, c+r)$ we obtain
$$\bar{v}(r,c+r) \geq \frac{1}{2r\Gamma(c)} \int_c^r \rho\overline{v}^2(\rho,\rho+c)d\rho. \tag{2.34}$$
Defining
$$\alpha(r) := \overline{v}(r,c+r), \ \ \beta(r) := \int_c^r \rho\alpha^2(\rho)d\rho \geq 0, \tag{2.35}$$
one can obtain, from (2.34), the following ODE type inequality
$$\beta'(r) = r\alpha^2(r) \geq \frac{1}{4r\Gamma^2}\beta^2(r) \geq 0, \ \text{for } r > c > R. \tag{2.36}$$
This is the crucial point, along the characteristic $(r, c+r)$, within Σ, one can study the solutions of the starting wave equation via this inequality. In particular, for the $K > 0$ case, one can find an upper bound for the time of blow-up via this argument, which was used to obtain (2.23).

7. In the $K = 0$ case, from this last inequality one can show, by a contradiction argument, that a global C^2 solution v of the IVP (2.27) must be zero along the characteristic $t = r + c$, for any $(r,t) \in \Sigma$ with $R < c < r$, concluding that $v(r,t) = 0$ for $(r,t) \in \Sigma$.

8. This fact, together with (2.31), allows us to conclude that in the region $\|x\| + t > R$, with $t > 0$, we have $v_t(x,t) = 0$. This implies, from the wave equation, that $\Delta v(x,t) = 0$ in this region.

9. Using $\Delta v(x,t) = 0$ and a fixed $t > R$, by the divergence theorem and the fact that $v(x,t)$ has compact support in x for a fixed t, we conclude that $v(x,t) = 0$ for $x \in \mathbb{R}^3$ and $t > R$.

10. The behavior of the solution in the remaining region $T_{0,R}$, i.e. the triangle with vertices
$$\{(-R,0),(0,R),(R,0)\}, \tag{2.37}$$
can be obtained by time-reversing the wave equation from $t = R$ to $t = 0$ and using again the uniqueness result, Corollary(2.2.1), showing that $v(x,t) = 0$ for $x \in \mathbb{R}^3$ and $t \geq 0$ if v is a global C^2 solution of $\Box v = (v_t)^2$.

In the following section, we show until what point we can recover the methods enumerated above and comment briefly on how to (possibly) proceed in order to obtain a proof of blow-up for (1.7) for small enough $p > 0$.

3 Main Problem

Our main goal is to understand the long time behavior of solutions to the following initial-value problem in 3+1 dimensions

$$\begin{cases} \Box v = -\frac{\gamma(p)}{t+1} v_t + (v_t)^2 =: \phi(x,t,v,v'), \ \text{ where } \ \gamma(p) = 2p/(1-p), \ \ p \in (0,1), \\ v|_{t=0}, \ v_t|_{t=0} \in C_0^\infty(\mathbb{R}^3). \end{cases} \tag{3.1}$$

As mentioned in the Introduction, this equation is obtained by considering (1.5) with the metric (1.4) for $p \in (0,1)$, together with a change of variable to conformal time.

The following is a summary of the ideas we can recover from the proof of Theorem 2.4.

1. Let $v \in C^2(\mathbb{R}^3 \times \mathbb{R}_+)$ be a solution of the IVP (3.1). The nonlinearity $\phi(x,t,v,v')$ is C^2 in its arguments and satisfies $\phi(x,t,0,0) = 0$. Thus, we are in the setting of the uniqueness theorem, Corollary 2.2.1, and we have $v(x,t) = 0$, for $\|x\| > t + R$, $t \geq 0$, where $R > 0$ is such that both *supp*$(v(x,0))$, and *supp*$(v_t(x,0))$ are contained in $B(0,R)$.

2. Considering the spherical average of the wave equation in (3.1), by the Euler-Poisson-Darboux equation we obtain
$$\Box(r\bar{v}) = -\frac{\gamma(p)}{t+1} r\bar{v}_t + r\overline{v_t^2}, \tag{3.2}$$
where we are again considering the symmetric extension of the spherical average.

3. The solution to the 1+1 dimensional wave equation (3.2), obtained by Duhamel's principle, is given by
$$\bar{v}(r,t) = \bar{v}^0(r,t) + \frac{1}{2r} \int\int_{T_{r,t}} \left(-\frac{\gamma(p)\rho}{\tau+1} \bar{v}_\tau(\rho,\tau) + \rho\overline{v_\tau^2}(\rho,\tau) \right) d\tau d\rho, \tag{3.3}$$
for $(r,t) \in \mathbb{R} \times \mathbb{R}_+$ where
$$\bar{v}^0(r,t) = \frac{1}{2r} \left[(r+t)\bar{v}(r+t,0) + (r-t)\bar{v}(r-t,0) + \frac{1}{2r} \int_{r-t}^{r+t} \rho\bar{v}_t(\rho,0) d\rho \right]. \tag{3.4}$$

4. Using the same region Σ considered before, due to parity of the integrand and symmetry, for $(r,t) \in \Sigma$ we have $\bar{v}^0(r,t) = 0$ and we can reduce the above to
$$\bar{v}(r,t) = \frac{1}{2r} \int\int_{T^*_{r,t}} \left(-\frac{\gamma(p)\rho}{\tau+1} \bar{v}_\tau(\rho,\tau) + \rho\overline{v_\tau^2}(\rho,\tau) \right) d\tau d\rho, \ \ (r,t) \in \Sigma. \tag{3.5}$$
Using Cauchy-Schwarz and commutation of time derivatives and the spherical average we can further write
$$\bar{v}(r,t) \geq \frac{1}{2r} \int\int_{T^*_{r,t}} \left(-\frac{\gamma(p)\rho}{\tau+1} \bar{v}_\tau(\rho,\tau) + \rho\bar{v}_\tau^2(\rho,\tau) \right) d\tau d\rho, \ \ (r,t) \in \Sigma. \tag{3.6}$$

5. From $(a-b)^2 \geq 0$ with $a = 1/(\tau+1)$ and $b = \bar{v}_\tau$ we get
$$-\frac{\bar{v}_\tau}{\tau+1} \geq -\frac{1}{2} \left(\bar{v}_\tau^2 + \frac{1}{(\tau+1)^2} \right), \tag{3.7}$$
so we can write (3.6) as
$$\bar{v}(r,t) \geq \frac{1}{2r} \left(1 - \frac{\gamma(p)}{2} \right) \int\int_{T^*_{r,t}} \rho\bar{v}_\tau^2 d\tau d\rho - \frac{\gamma(p)}{4r} \int\int_{T^*_{r,t}} \frac{\rho}{(\tau+1)^2} d\tau d\rho, \ \ (r,t) \in \Sigma, \tag{3.8}$$
where $c = t - r$ defines a characteristic chosen within Σ, see again Fig. 1.

6. To guarantee non-negativity of the left term on the right we choose, from this point onward, $1-\gamma(p)/2 \geq 0$, i.e. we restrict p such that $p \leq 1/2$. In this way the integral on the left can be restricted to $S_{r,t}$, as before. Thus, for $p \in [0,1/2]$, we have
$$\overline{v}(r,t) \geq \frac{1}{2r}\left(1-\frac{\gamma(p)}{2}\right)\int_c^r \int_{\rho-R}^{\rho+c} \rho \overline{v}_\tau^2 d\tau d\rho - \frac{\gamma(p)}{4r}\int\int_{T^*_{r,t}} \frac{\rho}{(\tau+1)^2} d\tau d\rho, \ \ (r,t) \in \Sigma. \tag{3.9}$$

7. Using the fundamental theorem of calculus and the fact that $\overline{v}(\rho,\rho-R)=0$ we have again
$$\int_{\rho-R}^{\rho+c} \overline{v}_\tau d\tau = \overline{v}(\rho,\rho+c), \tag{3.10}$$
which, using Cauchy-Schwarz's inequality again leads to
$$\overline{v}^2(\rho,\rho+c) = \left(\int_{\rho-R}^{\rho+c} \overline{v}_\tau d\tau\right)^2 \leq \left(\int_{\rho-R}^{\rho+c} 1 d\tau\right)\left(\int_{\rho-R}^{\rho+c} \overline{v}_\tau^2 d\tau\right), \tag{3.11}$$
and therefore
$$\int_{\rho-R}^{\rho+c} \overline{v}_\tau^2 d\tau \geq \frac{1}{\Gamma(c)}\left(\int_{\rho-R}^{\rho+c} \overline{v}_\tau d\tau\right)^2 = \frac{1}{\Gamma(c)}\overline{v}^2(\rho,\rho+c), \tag{3.12}$$
where again $\Gamma(c) = c+R > 0$.

8. Thus, for a fixed $(r,t) \in \Sigma$, along the characteristic defined by $c = t-r$ we can write
$$\overline{v}(r,c+r) \geq \frac{1}{2r\Gamma(c)}\left(1-\frac{\gamma(p)}{2}\right)\int_c^r \rho\overline{v}^2(\rho,\rho+c)d\rho - \frac{\gamma(p)}{4r}\int\int_{T^*_{r,c+r}} \frac{\rho}{(\tau+1)^2} d\tau d\rho. \tag{3.13}$$

All terms featuring $\gamma(p)$ are new compared to the equivalent expression (2.34), which naturally coincides at $p=0$. Rearranging terms to have non-negative quantities we get
$$\overline{v}(r,c+r) + \frac{\gamma(p)}{4r}\int\int_{T^*_{r,c+r}} \frac{\rho}{(\tau+1)^2} d\tau d\rho \geq \frac{1}{2r\Gamma(c)}\left(1-\frac{\gamma(p)}{2}\right)\int_c^r \rho\overline{v}^2(\rho,\rho+c)d\rho \geq 0, \tag{3.14}$$
for $r > c > R$ and $p \in [0,1/2]$. Defining $\alpha(r) := \overline{v}(r,c+r)$, $\beta(r) := \int_c^r \rho\overline{v}^2(\rho,\rho+c)d\rho$ and
$$f(r) := \frac{1}{4}\int\int_{T^*_{r,c+r}} \frac{\rho}{(\tau+1)^2} d\tau d\rho = \frac{r^2}{2} + \frac{cr}{2} - \frac{r}{2}\log(c+r+1), \tag{3.15}$$
we have as before $\beta'(r) = r\alpha^2(r) \geq 0$ and (3.14) becomes
$$\alpha(r) + \gamma(p)\frac{f(r)}{r} \geq \frac{1}{2r\Gamma}\left(1-\frac{\gamma(p)}{2}\right)\beta(r) \geq 0. \tag{3.16}$$
After some manipulations on (3.16) one finally obtains
$$\beta_p'(r) = r\alpha^2(r) \geq -\gamma(p)f^2(r) + \frac{1}{8\Gamma^2}\frac{(2-\gamma(p))^2}{r+\gamma(p)}\beta_p^2(r), \tag{3.17}$$
where the lower index p in β shows explicitly the dependence on p. Note that for $p=0$, implying $\gamma(p)=0$, we recover (2.36), as expected.

The main idea for the remainder of the argument is the following: we consider the initial value problem associated to the wave equation
$$\Box(r\overline{v}) = -\frac{\gamma(p)}{t+1} r\overline{v}_t + r\overline{v_t^2}, \tag{3.18}$$
for non-trivial initial data. If we prove continuity[3] in p, at $p=0$, for solutions in the neighborhood of some $(r^*, c+r^*) \in \Sigma$ where $\overline{v}_{p=0}(r^*) > 0$, due to the blow-up proof for (2.27), then we have $\beta_p(r^*) > 0$. One can then use (3.17) to prove blow-up of (3.1) for $0 < p < \epsilon$ for some $\epsilon > 0$.

[3]Possibly via the proof of local existence in $1+1$ dimensions.

References

[Chr20] Piotr T. Chruściel. *Elements of General Relativity.* Cham: Birkhäuser, 2020.

[CFO23] João L. Costa, Anne T. Franzen, and Jesús Oliver. "Semilinear Wave Equations on Accelerated Expanding FLRW Spacetimes". In: *Annales Henri Poincaré* 24.9 (2023). (`https://arxiv.org/abs/2201.05210`).

[Evans10] Lawrence C. Evans. *Partial differential equations.* Providence, R.I.: American Mathematical Society, 2010.

[FJ85] Fritz John. "Blow-Up for Quasi-Linear Wave Equations in Three Space Dimensions". In: *Fritz John: Collected Papers Volume 1.* Ed. by Jürgen Moser. Boston, MA: Birkhäuser Boston, 1985, pp. 566–588.

[FJ90] Fritz John. *Nonlinear Wave Equations, Formation of Singularities.* Vol. 2. University Lecture Series. American Mathematical Society, 1990.

[Lee19] J.M. Lee. *Introduction to Riemannian Manifolds.* Graduate Texts in Mathematics. Springer International Publishing, 2019. ISBN: 9783319917559.

[Luk] Jonathan Luk. *Introduction to Nonlinear Wave Equations.* (`https://web.stanford.edu/~jluk/NWnotes.pdf`).

[Ringström] Hans Ringström. *Non-linear wave equations.* (`https://people.kth.se/~hansr/nlw.pdf`).

[Sog95] C.D. Sogge. *Lectures on Nonlinear Wave Equations.* Monographs in analysis. International Press, 1995. ISBN: 9781571460325.